A two-grid finite-element/reduced basis scheme for the approximation of the solution of parameter dependent P.D.E

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Abstract

In the frame of optimization process in industrial framework, where numerical simulation is used at some stage, the same problem, modeled with partial differential equations depending on a parameter has to be solved many times for different sets of parameters. The reduced basis method may be successful in this frame and recent progress have permitted to make the computations reliable thanks to *a posteriori* estimators and to extend the method to non linear problems thanks to the "magic points" interpolation. However, it may not always be possible to use the code (for example of finite element type that allows for evaluating the elements of the reduced basis) to perform all the "off-line" computations required for an efficient performance of the reduced basis method. We propose here an alternating approach based on a coarse grid finite element the convergence of which is accelerated through the reduced basis and an improved post processing.

1 Introduction

Let X be a closed subspace of the Sobolev space $H^1(\Omega)$ over a bounded domain $\Omega \subset \mathbb{R}^d$ and $\mathcal{D}$ a set of parameter. We consider the following problem: given $\mu \in \mathcal{D}$, find $u(\mu) \in X$ such that

$$\forall v \in X, \quad a(u(\mu), v; \mu) = (f, v), \tag{1}$$

where a is a bilinear form, continuous and coercive over X that depends additionally on a parameter $\mu \in \mathcal{D}$ and $f \in L^2(\Omega)$.
In order to approximate the solution to this problem, one can use a standard numerical approach, as a finite element method, that provides a very accurate approximation of the solution $u(\mu)$ for any fixed value of the parameter μ. This accurate approximation will be called "truth approximation" in what follows. The computation of the truth approximation of $u(\mu)$ for many values of μ can however become very expensive as it has to be repeated for each parameter. The reduced basis method is an alternative that takes its roots upon the low

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"complexity" of the set of all solutions $\mathcal{M}^{\mathcal{D}} = \{u(\mu), \mu \in \mathcal{D}\}$ that can e.g. be measured by the Kolmogorov width [3] (see also [5]). This can for instance be formalized by the fact that for any $\epsilon > 0$, there exist a set of parameters $\mu_1, \mu_2, \cdots, \mu_N, \in \mathcal{D}$, where $N = N(\varepsilon)$ is reasonable, such that,

$$\forall \mu \in \mathcal{D} \quad \exists (\alpha_i(\mu)) \in \mathbb{R}^N, \qquad \|u(\mu) - \sum_{i=1}^{N} \alpha_i(\mu) u(\mu_i)\|_X \leq \varepsilon. \tag{2}$$

Based on the potential approximation property expressed above, the reduced basis method is in a Galerkin approach to the problem (1) for each new value of μ, within a space X_N spanned by N particular truth approximations of $u(\mu)$ corresponding to suitably chosen parameters μ.

To keep the interest of this method, i) the parameters μ_i have to be adequately chosen, ii) the corresponding solution has to be properly calculated or approximated through an accurate discretization method (as a finite element method, for example) and iii) moreover the construction of the stiffness matrix $A(\mu)$ with entries $a(u(\mu_i), u(\mu_j); \mu)$ as to be done for each new value of μ.

All the expensive computations involving in the three previous steps are done off-line which allows to have online computations that scales only like powers of N and do not involve the dimension of the finite element space (see e.g. [4]).

Various recent contributions have permitted to extend the range of the reduced basis method, e.g. the *a posteriori* error estimates for validation and determination of the proper parameters μ_i's [6], the "magic points" , for generalization to nonlinear problems [2].

In an industrial framework, for optimization processes for instance these approaches have a great potential, unfortunately part of the off line computations require to enter in the code that computes the truth approximation which is not possible in case the simulation code has been bought or relies on a long evolution so that only a black box use of the code is possible. Those computations require indeed the use of some component involved in the implementation of discretization method which are not available to the user. As a consequence the reduced basis method cannot be efficiently implemented, an alternative needs to be proposed.

2 An alternating reduced basis method

Let us assume that the truth approximation is based on a $\mathbb{P}_1$-finite element code, capable of giving us a good enough approximation of the $u(\mu)$ in a finite element space X_h such that

$$\forall \mu \in \mathcal{D}, \qquad ||u(\mu) - u_h(\mu)||_X \leq c_1 h < Tol. \tag{3}$$

Where Tol is a a tolerance chosen in accordance to the final goal we have. In the standard reduced basis method we first compute the truth approximation $u_h(\mu_i)$, then form a discrete space $X_h^N = \text{Span}\{u_h(\mu_i), i = 1, .., N\}$ and build a Garlerkin approximation of (1) in X_h^N: Find $u_h^N(\mu) \in X_h^N$ such that,

$\forall v \in X_h^N, \quad a(u_h^N(\mu), v; \mu) = (f, v)$. The implantation of this reduced basis method involves the construction of the stiffness matrix $A_h(\mu)$ with entries $a(u_h(\mu_i), u_h(\mu_j); \mu)$.

In an industrial framework, the finite element code is often locked, so we can not decompose the construction of the stiffness matrix $A_h(\mu)$ into a series of independent part that can be evaluated off line. This prevents us from employing the usual technique to compute quickly each stiffness matrix for a new value of μ, and take away the benefit of the reduced basis method (i.e. having a complexity depending only on N, independently of the dimension of the finite element space). First of all, let us remind that for a stable implementation of the reduced basis technique, it is required to build a better prepared basis than the one composed with the $u(\mu_i)$, usually a Gramm-Schmidt method is here advocated. We replace it here by the resolution of an eigenvalue problem: find $\xi \in X_h^N$ and $\lambda \in \mathbb{R}$ such that $\forall v \in X_h^N$, $\int_\Omega \nabla\xi\nabla v = \lambda \int_\Omega \xi v$, that provides $L^2(\Omega)$ and $H^1(\Omega)$ orthogonal eigenvectors $\xi_{i,BR}$ (chosen to be normalized in L^2). We note, that the $\xi_{i,BR}$ also constitute a second basis of the space X_h^N. Secondly we remark that, the standard reduced basis method aims at evaluating the coefficients intervening in the decomposition of $u_h^N(\mu)$ in the basis of the $\xi_{i,BR}$, $u_h^N(\mu) = \sum_{i=1}^N \beta_i^{BR}\xi_{i,BR}$. Those can appear as a substitute to the optimal coefficients $\beta_i^h(\mu) = \int_\Omega u_h(\mu)\xi_{i,BR}$ of the best approximation of $u_h(\mu)$ in X_h^N. This substitute is still good enough since, from Cea's Lemma we have $\|u(\mu) - u_h^N(\mu)\|_X \leq c \inf_{v\in X_h^N} \|u(\mu) - v\|_X$, then by using (2) and (3) we derive:

$$\|u(\mu) - u_h^N(\mu)\|_X \leq \varepsilon + c_2 h. \tag{4}$$

Our alternative method first presented in [1] and illustrated by numerical results proving the potential interest of this alternative consists in proposing, another surrogate to $\beta_i^h(\mu)$ defined by $\beta_i^H(\mu) = \int_\Omega u_H(\mu)\xi_{i,BR}$. Since, the computation of $u_H(\mu)$, for $H >> h$ and $X_H \subset X_h$, is less expensive than the one of $u_h(\mu)$, the use of the industrial code with the parameter H to construct the $\beta_i^H(\mu)$is cheap enough. From this computation we derive $u_N^{Hh}(\mu) = \sum_{i=1}^N \beta_i^H(\mu)\xi_{i,BR}$ in X_h^N. In what follow we explain in which case this can still be a very good approximation.

Since, $|\beta_i^h(\mu) - \beta_i^H(\mu)| \leq \|u_h(\mu) - u_H(\mu)\|_{0,\Omega}$ a classical Aubin-Nitsche argument [1] provides the following estimate: $\|u(\mu) - u_H(\mu)\|_{0,\Omega} \leq cH\|u(\mu) - u_H(\mu)\|_X \leq cH^2$. By using the L^2 and H^1 orthogonality of the $\xi_{i,BR}$ basis, we get that $\|u(\mu) - u_N^{Hh}(\mu)\|_X \leq \varepsilon + c_3 h + c_4 H^2$ where $c_4 = c_4(N)$ which is asymptotically similar to (4) when we choose $h \sim H^2$.

In the case of an higher order finite element approximation, $\mathbb{P}_k$, we can as well use an Aubin-Nitsche argument to get the improved error estimation. First we define $\Phi_{i,BR}$, such that $\forall v_h \in X_h$, $a(v_h, \Phi_{i,BR}; \mu) = \int_\Omega v_h \xi_{i,BR}$, hence

[1] Actually, the convergence results stated here either require that there is no corner or edge type singularities in the solutions — of the primal or dual problem for the Aubin-Nitsche argument — or that we relax somehow the definition of h and H being here a parameter associated with the grid size and the way the global refinement is done for convergence but not the size of the finer elements that should be defined such that the error bound by a conatant times h or H holds.

$\beta_i^h(\mu) - \beta_i^H(\mu) = \int_\Omega (u_h(\mu) - u_H(\mu))\xi_{i,h} = a(u_h(\mu) - u_H(\mu), \Phi_{i,h}; \mu)$. Since $X_H \subset X_h$ we obtain, from the definition of $u_h(\mu)$ and $u_H(\mu)$, that $\forall \chi_H \in X_H$, $a(u_h(\mu) - u_H(\mu), \chi_H; \mu) = 0$, then that $\forall \chi_H \in X_H$, $\beta_i^h(\mu) - \beta_i^H(\mu) = a(u_h(\mu) - u_H(\mu), \Phi_{i,h} - \chi_H; \mu)$.
Therefore we have $|\beta_i^h(\mu) - \beta_i^H(\mu)| \leq c\|u_h(\mu) - u_H(\mu)\|_X \|\Phi_h - \chi_H; \mu)\|_X \leq cH^{2k}$ and then
$\|u(\mu) - u_N^{Hh}(\mu)\|_X \leq \varepsilon + c_5 h^k + c_6 H^{2k}$ (where $c_6 = c_6(N)$). Finally, we get to the same conclusion as previously by choosing $h \sim H^2$. Here we want to improve even further the accuracy of the approach by proposing a simple post processing of the results.

3 Post-processing

Let $\underline{\beta}^H(\mu_j)$ be the vector $(\beta_i^H(\mu_j))_{1 \leqslant i \leqslant N}$ and $\underline{\beta}^h(\mu_j)$ the one corresponding to the $(\beta_i^h(\mu_j))_{1 \leqslant i \leqslant N}$. We decide to improve the computation of the $\underline{\beta}^H(\mu)$, by a post-processing that will insure that for each parameters $\mu = (\mu_j)_{j=1,\cdots,N}$ that are used in the construction of the reduced basis, the method returns exactly $u_h(\mu_j)$, indeed contrarily to $u_h(\mu)$ that we do not want to compute for a large number of values of μ, the truth solutions $u_h(\mu_j), j = 1, \cdots, N$ have been actually computed. In order to define this post-processing, we consider the linear transformation $\mathcal{F} : \mathbb{R}^N \to \mathbb{R}^N$, that maps $\underline{\beta}^H(\mu_j)$ on to $\underline{\beta}^h(\mu_j)$. The post processing consist in applying it to all the vector $\underline{\beta}^H(\mu)$.

Let T be the matrix associated to the transformation $\mathcal{F}$. For large values of N, the solutions $u_h(\mu_j), j = 1, \cdots, N$ mays become almost linearly dependent which results in a bad conditioning of the matrix T. This loss of stability may result in an important deterioration of the vectors $\underline{\beta}^H(\mu)$. To avoid this problem we propose to map only the first solutions in the previous set and thus construct an alternative matrix denoted T_k, $1 \leq k \leq N$ verifying:
$(T_k)\left(\underline{\beta}^H(\mu_{p_1}), \cdots, \underline{\beta}^H(\mu_{p_k}), \underline{\gamma}_{k+1}, \cdots, \underline{\gamma}_N\right)^t = \left(\underline{\beta}^h(\mu_{p_1}), \cdots, \underline{\beta}^h(\mu_{p_k}), \underline{\gamma}_{k+1}, \cdots, \underline{\gamma}_N\right)^t$
where the N vectors $\underline{\gamma}_k$ are constructed by a Gram - Schmidt method such that $\underline{\gamma}_1 = \frac{\underline{\beta}_H(\mu_{p_1})}{\|\underline{\beta}_H(\mu_{p_1})\|_2}$, and $\underline{\gamma}_k \in \text{Span}\{\underline{\beta}^H(\mu_{p_1}), \cdots, \underline{\beta}^H(\mu_{p_k})\}$.

The set $(\mu_{p_k})_{1 \leq k \leq N}$ is identical to the one used in the construction of the reduced basis, but it has been arranged differently. Indeed for each iteration k, we choose μ_{p_k} among the $N - k$ parameters $(\mu_{p_q})_{,k \leq q \leq N}$, such that $\max_{1 \leq j \leq N} \|T_k \beta^H(\mu_j) - \beta^h(\mu_j)\|_\infty$ is the smallest. We notice that, at the end, the matrix T_N and T are similar.

Then we chose the matrix T_k with the largest k chosen in such a way that the condition number of the matrix T_k is moderate enough.

4 Numerical results

The problems we consider in this section are in 2 dimensions. From an original coarse triangulation $\mathcal{T}_{H_0}$, we built successive refined triangulations $\mathcal{T}_{H_i, 1 \leq i \leq 4}$ by recursively splitting each triangle K in $\mathcal{T}_{H_{i-1}}$ into four triangles with equal

diameter H_{i_K} such that $H_{i_K} = \frac{H_{(i-1)K}}{2}$. We get a superspace X_{H_i} about four times larger than $X_{H_{i-1}}$ that satisfies $X_{H_0} \subset X_{H_n}$. We denote by $u_N^{hP}(\mu)$, the H^1 projection of $u_h(\mu)$ on the basis of the $\xi_{i,BR}$, defined by $u_N^{hP}(\mu) = \sum_{i=1}^{N} \beta_i^h(\mu)\xi_{i,BR}$. It is the best we can expect from the reduced basis, that is one of the ingredient entering in the approximation.

4.1 Example 1

We first consider the nonlinear problem: find $u \in H^1(\Omega)$ such that

$$
\begin{array}{l}
-\Delta u + u^3 = sin(x)sin(y) \text{ in } \Omega = [0,1]^2 \setminus (]\frac{1}{2},1[^2) \textit{ (L-shape domain)} \\
\alpha u + \frac{\partial u}{\partial n} = y(1-y) \text{ on } \Gamma_F = \{(1,y), y \in [0,\frac{1}{2}]\} \\
u = \eta\ xy(1-y)(1-x) \text{ on } \Gamma_D = \partial\Omega \setminus \Gamma_F
\end{array}
$$

In this example, the set of parameters, $\mu = (\alpha, \eta)$, that we use is varying in $[1,37] \times [1,100]$. Let be $\mu_{H_i} = \underset{\mu=(\beta,\eta)\in[1,37]\times[1,100]}{\text{argmax}} \{||u(\mu) - u_N^{hH_i}(\mu)||_{1,\Omega}\}$ and $\mu_h = \underset{\mu=(\beta,\eta)\in[1,37]\times[1,100]}{\text{argmax}} \{||u(\mu) - u^h(\mu)||_{1,\Omega}\}$.

The $\mathbb{P}_1$ approximation results's are showed in the table 1. We first remark that we need at least $N = 10$ elements in the reduced basis to recover the truth error. Second, before post-processing we note that the H_1-error made with the solution $u_N^{hH_2}$ is close to the one made with u_h, for any value of μ in $\mathcal{D}$, despite the fact that $\mathcal{T}_{H_2}$ is eight times less accurate than $\mathcal{T}_{H_4}$, at least for $N \geq 10$. We also note a small deterioration of the evaluation of the solution, when N rises, confirming that the constant $c_4(N)$ is growing with N. Finally, we note that the post-processing improved even more the approaximation since it allows to recover the truth error even starting from the computations of the coarsest solution $u_N^{hH_0}$, at least if we use the proper number of reduced elements (10 or 15) , which is a very substantial savings. We note also that the reduction of indices in the post-processing is used, even it is important since, in the case $N = 15$ the error $||u(\mu_{H_1}) - u_N^{hH_1}(\mu_{H_1})||_{1,\Omega}$ with the full matrix is 0.50.

4.2 Exemple 2

The second problem is a convection dominated problem : find $u \in H^1(\Omega)$ such that

$$
\begin{array}{l}
(0.01)\Delta u + v \cdot \nabla u = 0 \text{ in}\Omega = [0,1]^2 \\
u = x^2 \text{ on } \Gamma_1 = \{(1,y), y \in [0,1]\} \\
u = y^2 \text{ on } \Gamma_2 = \{(x,1), x \in [0,1]\} \\
u = 0 \text{ on } \Gamma_3 = \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2).
\end{array}
$$

where v is such as $v = (\cos\ \mu,\ \sin\ \mu)$. Here, the varying parameter is the angle of the convection $\mu \in [0, \frac{\pi}{2}]$. Let be $\mu_{H_i} = \underset{\mu\in[0,\frac{\pi}{2}]}{\text{argmax}}\{||u(\mu) - u_N^{hH_i}(\mu)||_{1,\Omega}\}$ and $\mu_h = \underset{\mu\in[0,\frac{\pi}{2}]}{\text{argmax}}\{||u(\mu) - u^h(\mu)||_{1,\Omega}\}$. The table 3 shows the $\mathbb{P}_1$ approximations result's, while the table 2 shows the $\mathbb{P}_2$ ones. We can make the same conclusion than in the previous example : this combined method (reduced bais + two grids)

is thus even improved by the trivial postprocessing. Note that the mathematical justification of this last ingredient is still missing.

Table 1: Error for the example 1 with $X_h = \{v \in \mathcal{C}^0(\overline{\Omega}),\ v_{|T} \in \mathbb{P}_1(T),\ T \in \mathcal{T}_{H_4}\}$

$$||u(\mu_h) - u_h(\mu_h)||_{1,\Omega} = 3.3 \times 10^{-2}$$

N	k	i	$\|\|u(\mu_{H_i}) - u_N^{hH_i}(\mu_{H_i})\|\|_{1,\Omega}$ with post-processing	$\|\|u(\mu_{H_i}) - u_N^{hH_i}(\mu_{H_i})\|\|_{1,\Omega}$ without post-processing	$\|\|u(\mu_{H_i}) - u_{H_i}(\mu_{H_i})\|\|_{1,\Omega}$
5	$\|\|u(\mu_h) - u_h^{BR}(\mu_h)\|\|_{1,\Omega} = 0.19$				
	5	0	0.15	0.13	0.49
		1	0.17	0.16	0.28
		2	0.19	0.18	0.15
		3	0.19	0.19	7.3×10^{-2}
10	$\|\|u(\mu_h) - u_h^{BR}(\mu_h)\|\|_{1,\Omega} = 3.6 \times 10^{-2}$				
	10	0	3.6×10^{-2}	0.35	0.49
		1	3.5×10^{-2}	6.8×10^{-2}	0.28
		2	3.5×10^{-2}	3.8×10^{-2}	0.15
		3	3.6×10^{-2}	3.5×10^{-2}	7.3×10^{-2}
15	$\|\|u(\mu_h) - u_h^{BR}(\mu_h)\|\|_{1,\Omega} = 3.4 \times 10^{-2}$				
	15	0	3.5×10^{-2}	0.47	0.49
	7	1	3.7×10^{-2}	0.14	0.28
	15	2	3.4×10^{-2}	3.4×10^{-2}	0.15
		3	3.4×10^{-2}	3.4×10^{-2}	7.3×10^{-2}
20	$\|\|u(\mu_h) - u_h^{BR}(\mu_h)\|\|_{1,\Omega} = 3.3 \times 10^{-2}$				
	20	0	5.5×10^{-2}	0.56	0.49
		1	3.4×10^{-2}	0.20	0.28
		2	3.4×10^{-2}	4.8×10^{-2}	0.15
		3	3.4×10^{-2}	3.4×10^{-2}	7.3×10^{-2}

Table 2: Error for the example 2 with $X_h = \{v \in \mathcal{C}^0(\overline{\Omega}),\ v_{|T} \in \mathbb{P}_2(T),\ T \in \mathcal{T}_{H_3}\}$

$$\|u(\mu_h) - u_h(\mu_h)\|_{1,\Omega} = 3.5 \times 10^{-3}$$

N	k	i	$\|\|u(\mu_{H_i}) - u_N^{hH_i}(\mu_{H_i})\|\|_{1,\Omega}$ with post-processing	$\|\|u(\mu_{H_i}) - u_N^{hH_i}(\mu_{H_i})\|\|_{1,\Omega}$ without post-processing	$\|\|u(\mu_{H_i}) - u_{H_i}(\mu_{H_i})\|\|_{1,\Omega}$
5	$\|\|u(\mu_h) - u_h^{BR}(\mu_h)\|\|_{1,\Omega} = 1.41 \times 10^{-2}$				
	5	0	6.4×10^{-2}	6.5×10^{-2}	0.11
		1	6.1×10^{-2}	6.1×10^{-2}	3.3×10^{-2}
		2	6.1×10^{-2}	6.1×10^{-2}	8.7×10^{-3}
10	$\|\|u(\mu_h) - u_h^{BR}(\mu_h)\|\|_{1,\Omega} = 3.5 \times 10^{-3}$				
	10	0	3.5×10^{-3}	4.1×10^{-2}	0.16
		1	3.5×10^{-3}	5.6×10^{-3}	5.1×10^{-2}
		2	3.5×10^{-3}	3.5×10^{-3}	1.4×10^{-2}
15	$\|\|u(\mu_h) - u_h^{BR}(\mu_h)\|\|_{1,\Omega} = 3.5 \times 10^{-3}$				
	15	0	3.5×10^{-3}	5.8×10^{-2}	0.16
		1	3.5×10^{-3}	7.8×10^{-3}	5.1×10^{-2}
		2	3.5×10^{-3}	3.5×10^{-3}	01.4×10^{-2}

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Table 3: Error for the example 2 with $X_h = \{v \in \mathcal{C}^0(\overline{\Omega}),\, v_{|T} \in \mathbb{P}_1(T),\, T \in \mathcal{T}_{H_4}\}$

$$\|u(\mu_h) - u_h(\mu_h)\|_{1,\Omega} = 3.5 \times 10^{-2}$$

N	k	i	$\|\|u(\mu_{H_i}) - u_N^{hH_i}(\mu_{H_i})\|\|_{1,\Omega}$ with post-processing	$\|\|u(\mu_{H_i}) - u_N^{hH_i}(\mu_{H_i})\|\|_{1,\Omega}$ without post-processing	$\|\|u(\mu_{H_i}) - u_{H_i}(\mu_{H_i})\|\|_{1,\Omega}$
5	$\|\|u(\mu_h) - u_h^{BR}(\mu_h)\|\|_{1,\Omega} = 3.6 \times 10^{-2}$				
	5	0	5.7×10^{-2}	0.17	0.45
		1	6.2×10^{-2}	9.0×10^{-2}	0.24
		2	6.5×10^{-2}	7.0×10^{-2}	0.12
		3	6.6×10^{-2}	6.7×10^{-2}	6.1×10^{-2}
10	$\|\|u(\mu_h) - u_h^{BR}(\mu_h)\|\|_{1,\Omega} = 3.5 \times 10^{-2}$				
	10	0	3.5×10^{-2}	0.244	0.53
		1	3.5×10^{-2}	9.1×10^{-2}	0.31
		2	3.5×10^{-2}	4.2×10^{-2}	0.16
		3	3.5×10^{-2}	3.6×10^{-2}	7.9×10^{-2}
15	$\|\|u(\mu_h) - u_h^{BR}(\mu_h)\|\|_{1,\Omega} = 3.5 \times 10^{-2}$				
	15	0	3.5×10^{-2}	0.36	0.53
		1	3.5×10^{-2}	9.8×10^{-2}	0.31
		2	3.5×10^{-2}	4.2×10^{-2}	0.16
		3	3.5×10^{-2}	3.6×10^{-2}	7.9×10^{-2}
20	$\|\|u(\mu_h) - u_h^{BR}(\mu_h)\|\|_{1,\Omega} = 3.5 \times 10^{-2}$				
	13	0	3.5×10^{-2}	0.37	0.53
		1	3.5×10^{-2}	0.13	0.31
		2	3.5×10^{-2}	4.6×10^{-2}	0.16
		3	3.5×10^{-2}	3.6×10^{-2}	7.9×10^{-2}